Inequalities
https://www.linkedin.com/groups/8313943/8313943-6396576505849274371
Let x, y, z be non-negative reals such that $x^2 + +y^2 + z^2 + xyz = 4$.
Show that $xyz \le xy + yz + zx \le xyz + 2$.
Solution by Arkady Alt, San Jose,California, USA.
First note that by AM-GM inequality $4 - xyz = x^2 + y^2 + z^2 \ge 3\left(x^2y^2z^2\right)^{1/3} \iff$

$3\left(xyz\right)^{2/3} + xyz - 4 \le 0 \iff \left(\left(xyz\right)^{1/3} - 1\right)\left(\left(xyz\right)^{1/3} + 2\right)^2 \le 0 \iff$
$xyz \le 1$.

Then $xy + yz + zx \ge 3\left(x^2y^2z^2\right)^{1/3}$ and since $\left(xyz\right)^{2/3} \ge xyz$ we obtain $xy + yz + zx \ge 3xyz \ge xyz$.

Also note that inequality $xy + yz + zx \le xyz + 2$ holds if at least one of variables x, y, z
equal to zero. Indeed, let $z = 0$.Then $x^2 + y^2 = 4$ and,therefore, $xy \le \dfrac{x^2 + y^2}{2} = 2$.

Thus, remains to prove inequality $xy + yz + zx \le xyz + 2$ for $x, y, z > 0$.

Since all positive solutions of equation $x^2 + y^2 + z^2 + xyz = 4$ can be represented in the
form $x = 2\cos\alpha, y = 2\cos\beta, z = 2\cos\gamma$,where $\alpha, \beta, \gamma \in (0, \pi/2)$ and $\alpha + \beta + \gamma = \pi$ then
$xy + yz + zx \le xyz + 2$ becomes $4\sum\cos\alpha\cos\beta \le 8\cos\alpha\cos\beta\cos\gamma + 2 \iff$
(1) $\quad 2\sum\cos\alpha\cos\beta \le 4\cos\alpha\cos\beta\cos\gamma + 1$.

Let ABC be some triangle with angles α, β, γ and R, r, s be,respectively, circumradius,inradius
and semiperimeter of $\triangle ABC$.Then, since $\cos\alpha\cos\beta\cos\gamma = \dfrac{s^2 - (2R + r)^2}{4R^2}$ and
$\sum\cos\alpha\cos\beta = \dfrac{s^2 + r^2 - 4R^2}{2R^2}$ inequality **(1)** becomes
$\dfrac{s^2 + r^2 - 4R^2}{2R^2} \le \dfrac{s^2 - (2R + r)^2}{R^2} + 1 \iff s^2 + r^2 - 4R^2 \le 2\left(s^2 - (2R + r)^2\right) + 2R^2 \iff 2\left(s^2 - (2R + r)^2\right) + 2R^2 - \left(s^2 + r^2 - 4R^2\right) = -2R^2 - 8Rr - 3r^2 + s^2$
$2R^2 + 8Rr + 3r^2 \le s^2$ (**Walker's Inequality*for acute angled triangle).**

My proof of the **Walker's Inequality.**

First we will prove that in any acute angled triangle holds inequality $a^2 + b^2 + c^2 \ge 4\left(R + r\right)^2$.

We have $a^2 + b^2 + c^2 \ge 4\left(R + r\right)^2 \iff 4R^2\left(\sin^2 A + \sin^2 B + \sin^2 C\right) \ge 4R^2\left(1 + \dfrac{r}{R}\right)^2 \iff$
$\sin^2 A + \sin^2 B + \sin^2 C \ge \left(\cos A + \cos B + \cos C\right)^2 \iff$
$\sum_{cyc}\left(1 - \cos^2 A\right) \ge \sum_{cyc}\cos^2 A + 2\sum_{cyc}\cos B\cos C \iff \sum_{cyc}\left(\cos A + \cos B\right)^2 \le 3$.

Since by Cauchy Inequality

$(\cos A+\cos B)^2 \leq (a\cos B+b\cos A)\left(\dfrac{\cos B}{a}+\dfrac{\cos A}{b}\right)=c\left(\dfrac{\cos B}{a}+\dfrac{\cos A}{b}\right)$

Then $\sum\limits_{cyc}(\cos A+\cos B)^2 \leq \sum\limits_{cyc}\left(\dfrac{c\cos B}{a}+\dfrac{c\cos A}{b}\right)=\dfrac{c\cos B}{a}+\dfrac{c\cos A}{b}+\dfrac{a\cos C}{b}+$

$\dfrac{a\cos B}{c}+\dfrac{b\cos A}{c}+\dfrac{b\cos C}{a}=\sum\limits_{cyc}\left(\dfrac{c\cos B+b\cos C}{a}\right)=\sum\limits_{cyc}\dfrac{a}{a}=3.$

Noting that $ab+bc+ca=s^2+4Rr+r^2$ we obtain

$4s^2=(a+b+c)^2=a^2+b^2+c^2+2(ab+bc+ca)\geq 4(R+r)^2+2\left(s^2+4Rr+r^2\right)=$
$4R^2+16Rr+6r^2+2s^2 \iff s^2\geq 2R^2+8Rr+3r^2.$